An Inverse Finite Element Method for Application to Structural Health Monitoring

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14th U.S. National Congress of Theoretical and Applied Mechanics
June 27, 2002

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Motivation

- Next generation of aerospace/aircraft vehicles
  - Multifunctional structures
  - Fiber Optics (FO) sensor system
  - Structural health monitoring in real time

- Morphing wing technology
  - deformations by embedded actuators

- "Joined Wing" aircraft
  - Radar on wing surfaces
  - Wing deflections
  - Radar adjustment
Inverse Problem

- Strain sensors on surfaces of structural components
- Arbitrary location and orientation

Objective:
Determine deformed shape of structure in real time
Approach

Develop *Inverse FEM* for reconstruction of *deformation* using measured strain data, i.e., integrate $\varepsilon-u$ relations to obtain $u$.
Background

- Tikhonov-Arsenin (1977)
  - Ill-posed, inverse problems, regularization method
- Tessler-Dong (’81), Tessler-Hughes (’85)
  - Anisoparametric $C^0$ elements
- Tessler et al (’93)
  - Discrete least-squares finite elements for smoothing of data
Variational Formulation of Inverse FEM Element level

Find an extremum of the smoothing functional for a fixed value of the regularization parameter $\lambda$:

$$
\Phi^\lambda(\mathbf{u}^h) = \|\mathbf{\varepsilon}(\mathbf{u}^h) - \mathbf{\varepsilon}^\delta\|^2 + \|\mathbf{\kappa}(\mathbf{u}^h) - \mathbf{\kappa}^\delta\|^2 + \lambda \|\mathbf{\gamma}(\mathbf{u}^h) - \mathbf{\gamma}^\delta\|^2
$$

$$
\|\mathbf{\varepsilon}(\mathbf{u}^h) - \mathbf{\varepsilon}^\delta\|^2 = \frac{1}{n} \sum_{i=1}^{n} \left[ \mathbf{\varepsilon}(\mathbf{u}^h)_{x_i} - \mathbf{\varepsilon}^\delta_i \right]^2
$$

Euclidean squared norm in terms of membrane strains

$$
\|\mathbf{\kappa}(\mathbf{u}^h) - \mathbf{\kappa}^\delta\|^2 = \frac{\Omega^e}{n} \sum_{i=1}^{n} \left[ \mathbf{\kappa}(\mathbf{u}^h)_{x_i} - \mathbf{\kappa}^\delta_i \right]^2
$$

Norm in terms of bending curvatures

$$
\|\mathbf{\gamma}(\mathbf{u}^h) - \mathbf{\gamma}^\delta\|^2 = \frac{1}{n} \sum_{i=1}^{n} \left[ \mathbf{\gamma}(\mathbf{u}^h)_{x_i} - \mathbf{\gamma}^\delta_i \right]^2
$$

Norm in terms of transverse shear strains

$\mathbf{\varepsilon}^\delta_i, \mathbf{\kappa}^\delta_i, \mathbf{\gamma}^\delta_i$

Arrays of discrete measured strains at $x_i$
**Strain-displacement relations**

\[
\epsilon = \begin{bmatrix}
\epsilon_{xo} \\
\epsilon_{yo} \\
\gamma_{x yo}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial}{\partial x} & 0 & 0 & 0 & 0 \\
0 & \frac{\partial}{\partial y} & 0 & 0 & 0 \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
u \\
v \\
w \\
\theta_x \\
\theta_y
\end{bmatrix}
\]

\[
u^h \equiv N d \quad \text{C}^0 \text{ interpolated displacements}
\]

\[
\kappa = \begin{bmatrix}
\kappa_{xo} \\
\kappa_{yo} \\
\kappa_{x yo}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & \frac{\partial}{\partial x} & 0 \\
0 & 0 & \frac{\partial}{\partial y} & 0 & 0 \\
0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0
\end{bmatrix} \begin{bmatrix}
u \\
v \\
w \\
\theta_x \\
\theta_y
\end{bmatrix}
\]

\[
\gamma = \begin{bmatrix}
\gamma_{xzo} \\
\gamma_{yzo}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & \frac{\partial}{\partial x} & 0 & 1 \\
0 & 0 & \frac{\partial}{\partial y} & 1 & 0
\end{bmatrix} \begin{bmatrix}
u \\
v \\
w \\
\theta_x \\
\theta_y
\end{bmatrix}
\]

**Measured strains**

\[
\epsilon_{xo}^\delta = \frac{1}{2} (\epsilon_{xx}^+ + \epsilon_{xx}^-),
\]

\[
\kappa_{xo}^\delta = \frac{1}{t} (\epsilon_{xx}^+ - \epsilon_{xx}^-)
\]

[Diagram of strain gauge]
Linear Equations

Minimize element smoothing functional

$$\Phi^\lambda(u^h) = \|\varepsilon(u^h) - \varepsilon^\delta\|^2 + \|\kappa(u^h) - \kappa^\delta\|^2 + \lambda \|\gamma(u^h) - \gamma^\delta\|^2$$

And summing on all elements results in

$$Kd = F \quad \text{Ultra-fast solution}$$

$$K \equiv K(x_i) \quad \text{Symmetric and positive definite}$$

$$d \equiv d(u) \quad \text{Vector of displacement dof's}$$

$$F \equiv F(\varepsilon^\delta) \quad \text{R.h.s. "load" vector}$$
MIN3 and its Inverse Element, MIN3⁻¹

**Direct FEM: MIN3**, 3-node plate ('85)
- C⁰ kinematic interpolations
  - u, v, θₓ, θᵧ -- linear
  - w -- anisoparametric quadratic
  - Membrane and bending strains constant

**Inverse FEM: MIN3⁻¹**, 3-node element
- C⁰ kinematic interpolations as MIN3
- Least squares formulation:
  - Minimize: Φ
  - Loads and materials unspecified

**Principle of Min. Potential Energy:**
- Minimize: Π
- φ² K_{shear} (MIN3*)

[Diagram of triangle element with degrees of freedom (dof) u, v, w, θₓ, θᵧ and Ωₑ]
Inverse FEM: Mapping of strain data

Arbitrary distribution of strain sensors on Idealized Wing model

Mapping I: 1-to-1

Mapping II: n-to-1
Numerical Experiment: Idealized Wing Model

- Aluminum panel clamped at left end
- Loads
  - Uniform pressure
  - Twisting forces
  - In-plane forces
- Ultra-thin plate: span/thickness=6*10^4
FEM (MIN3*):
Convergence of Deflection and Bending Rotation

※ MIN3* strains will be used to represent experimental data
Low-Fidelity FEM (MIN3\textsuperscript{*}) Strain Distributions, i.e.
Low Quality "Experimental Data"
MIN3\(^{-1}\) Displacement Reconstruction from MIN3\(^*\) strain data

**Mapping l (1-1)**

**FEM (MIN3\(^*\)) strains**

**MIN3\(^{-1}\) discretization**

**Results @ node A**

\[
\lambda = 10^{-6}
\]

\[
100 \frac{(w - w_{inv})}{w}
\]

%Error in w

dof

\[
\lambda = 10^{-6}
\]

\[
100 \frac{(w - w_{inv})}{w}
\]

**MF**

**HF**

**LF**

All other displacement variables reconstructed precisely across panel
High-Fidelity FEM (MIN3*) Strain Distributions. i.e. Accurate "Experimental Data"
Study of Lambda Parameter

Mapping I (1-1)

Results @ node A

High Fidelity

FEM (MIN3*) strains

High Fidelity

MIN3⁻¹ discretization

* In-plane displacements, u and v, reconstructed precisely across panel
Study of Lambda Parameter

Mapping II (n-1)

High Fidelity

FEM(MIN3*) strains

Low Fidelity

Inverse FEM (MIN3⁻¹)

Results @ node A

\[ \varepsilon^\delta = \varepsilon^* (1 + 0.05 \times \delta), \]
\[ \delta \in [-1, 1] \text{ (pseudorandom number)} \]
Remarks on Numerical Results

- MIN3\(^{-1}\) is exact inverse of MIN3
  - In 1-1 mapping of MIN3 strains, MIN3\(^{-1}\) reconstructs all displacements precisely

- MIN3* more accurate in bending than MIN3

- Shear locking eliminated via anisoparametric interpolation

- Mapping I (1-1)
  - Effect of Lambda is insignificant

- Mapping II (n-1)
  - Serves as filter/smooth
  - "Optimal" Lambda
  - Strains with random error "smoothed" effectively
Concluding Remarks

- Standard FEM architecture
- Arbitrary topology application
- Static and dynamic problems
- Small and large displacements (incremental linear approach)
- Thin and thick beams, plates, shells, solids and built-up structures
- Sensor location optimization
- Ultra-fast computations