An Inverse Finite Element Method for Reconstruction of Elastic Deformations in Beam, Plate and Shell Structures

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Structural health monitoring

- Civil and aeronautical structures
- Fiber optic Bragg-grating sensors measure
  - Strains
  - Temperatures
- Full-field quantities of interest (in real time)
  - Displacements (morphing wing, radar antenna)
  - Strains
  - Stresses
  - Failure criteria
Objective

- Develop fast and robust algorithm for reconstruction of full-field displacements from measured strains – an inverse problem
Approach

- Minimize Error Functional (variational principle)
- Inverse Boundary Value Problem
- Inverse FEM (Strain Integration Algorithm)
Outline

- Inverse reconstruction problem
- First-order shear deformation theory
- Penalty error functional
- Inverse finite element method
- Computational validation
- Computational-experimental validation
- Summary
Inverse reconstruction problem

- Inverse problems are ill-posed
  - Do not satisfy conditions of existence, uniqueness, and stability
- Uniqueness
  - various strains and boundary conditions that correspond to nearly same displacements
- Instability
  - Small disturbances in measured data (strains) cause great changes in solution (displacements)
- Tikhonov's regularization method
  - Improves robustness of algorithms by enforcing additional physical constraints ensuring smoothness of solution
First-order shear deformation theory

- Displacements

\[ u_x(x, y, z) = u + z \theta_y \]
\[ u_y(x, y, z) = v + z \theta_x \]
\[ u_z(x, y, z) = w \]
Strain-displacement relations

- In-plane strains

\[
\begin{align*}
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\gamma_{xy}
\end{bmatrix} &= \begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\gamma_{xy}
\end{bmatrix} + \begin{bmatrix}
K_{xx} \\
K_{yy} \\
K_{xy}
\end{bmatrix} \\
&= \begin{bmatrix}
\frac{\partial^2}{\partial x^2} \\
\frac{\partial^2}{\partial y^2} \\
\frac{\partial^2}{\partial x \partial y}
\end{bmatrix} \begin{bmatrix}
u \\
v \\
w \\
\theta_x \\
\theta_y
\end{bmatrix} \\
&= \begin{bmatrix}
0 & 0 & 0 & \frac{\partial^2}{\partial x} & 0 \\
0 & 0 & \frac{\partial^2}{\partial y} & 0 & 0 \\
0 & 0 & 0 & \frac{\partial^2}{\partial x \partial y} & 0 \\
\frac{\partial^2}{\partial x} & \frac{\partial^2}{\partial y} & \frac{\partial^2}{\partial x \partial y}
\end{bmatrix}
\end{align*}
\]

- Transverse shear strains

\[
\gamma = \begin{bmatrix}
\gamma_{xzo} \\
\gamma_{yzo}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & \frac{\partial^2}{\partial x} & 0 & 1 \\
0 & 0 & \frac{\partial^2}{\partial y} & 1 & 0
\end{bmatrix}
\]

Measured strains

\[
\begin{align*}
\mathbf{\varepsilon}_{xo} & \delta = \frac{1}{2} \left( \mathbf{e}_{xx}^+ + \mathbf{e}_{yy}^- \right) \\
\mathbf{\varepsilon}_{yo} & \quad \mathbf{\gamma}_{xyo} = \frac{1}{2} \left( \mathbf{e}_{xx}^+ - \mathbf{e}_{yy}^- \right) \\
\mathbf{k}_{xo} & \delta = \frac{1}{2t} \left( \mathbf{e}_{xx}^+ - \mathbf{e}_{yy}^- \right) \\
\mathbf{k}_{yo} & \quad \mathbf{k}_{xyo} = \frac{1}{2t} \left( \mathbf{e}_{xx}^+ \right) \\
\end{align*}
\]

In thin plates "measured" shear strains can be omitted

Top strain gauge

Bottom strain gauge
Error smoothing functional

- Find an extremum of the smoothing functional for a fixed value of penalty parameter \( \lambda \)

\[
\Phi^\lambda(u^h) = \|\varepsilon(u^h) - \varepsilon^\delta\|^2 + \|\kappa(u^h) - \kappa^\delta\|^2 + \lambda \|\gamma(u^h) - \gamma^\delta\|^2
\]

Euclidean squared norms

\[
\|\varepsilon(u^h) - \varepsilon^\delta\|^2 = \frac{1}{n} \sum_{i=1}^{n} \left(\varepsilon(u^h)_{x_i} - \varepsilon^\delta_{x_i}\right)^2
\]

\[
\|\kappa(u^h) - \kappa^\delta\|^2 = \frac{\Omega_e}{n} \sum_{i=1}^{n} \left[\kappa(u^h)_{x_i} - \kappa^\delta_{x_i}\right]^2
\]

\[
\|\gamma(u^h) - \gamma^\delta\|^2 = \frac{1}{n} \sum_{i=1}^{n} \left[\gamma(u^h)_{x_i} - \gamma^\delta_{x_i}\right]^2
\]

- Weights in error functional \( \{1, 1, \lambda\} \)

- 2nd and 3rd terms coupled

- Generally \( \lambda \ll 1 \)

\( \varepsilon_i^\delta, \; \kappa_i^\delta, \; \gamma_i^\delta \) Arrays of measured strains at \( x_i \)
Special case: Thin plates and shells

- 3rd term

\[ \lambda \| \gamma(u^h) - \gamma^\delta \|^2 = \frac{\lambda}{\Omega^e} \int_{\Omega^e} \gamma(u^h)^2 \, dx \, dy \]

- Kirchhoff constraints enforced

\[
\begin{bmatrix}
\gamma_{xzo} \\
\gamma_{yzo}
\end{bmatrix} \equiv \begin{bmatrix}
w_x + \theta_y \\
w_y + \theta_x
\end{bmatrix} \to 0
\]
Kinematic interpolations for inverse flat shell element (MIN3⁻¹)

\[ \mathbf{u}^h \equiv \{ u, v, w, \theta_x, \theta_y \} = N \mathbf{d} \]

- \( u \): linear
- \( v \): linear
- \( w \): quadratic
- \( \theta_x \): linear
- \( \theta_y \): linear

**u^h \equiv N d** anisoparametric

**N:** \( C^0 \) - continuous shape functions

**d:** displacement dof's

\[ \{ \varepsilon_{x0}, \varepsilon_{y0}, \gamma_{xy0} \} \text{ constant} \]

\[ \{ K_{x0}, K_{y0}, K_{xy0} \} \text{ constant} \]

\[ \{ \gamma_{xzo}, \gamma_{yzo} \} \text{ linear} \]

Tessler-Hughes, CMAME (1985)
Inverse FEM equations

- Minimize sum of element contributions

\[ \delta \left[ \sum_{e=1}^{N} \Phi_e \lambda(u^h) \right] = 0 \]

results in

\[ d = K^{-1} F \]

\( K \equiv K(x_i) \) Symmetric, positive definite
\( d \equiv d(u) \) Displacement dof's
\( F \equiv F(\varepsilon^{\delta}) \) r.h.s. vector

Fast computation
Computational validation

- Direct FEM solution
  - Mesh, materials, loads and B.C.'s
  - Output strains at optimal points
- Inverse FEM solution
  - Direct analysis strains used as "experimental" strains
  - "Experimental" strains mapped onto inverse FEM mesh
  - Apply same kinematic B.C.'s as in direct analysis
  - Solve inverse FEM equations to obtain displacements
- Comparison
  - reconstructed displacements vs. displacements from direct analysis
Inverse FEM: Mapping of strains

Arbitrary distribution of strain sensors on Wing model

1-to-1

Inverse FEM mesh

Element centroid

n-to-1
Idealized wing: Direct FE analysis

- Clamped AL panel
- Loads
  - Uniform pressure
  - Twisting forces
  - In-plane forces
- Span / thickness = 6*10^4

Direct FEM

<table>
<thead>
<tr>
<th>Element</th>
<th>Interpolations</th>
</tr>
</thead>
<tbody>
<tr>
<td>MIN3</td>
<td>$u^h$</td>
</tr>
</tbody>
</table>

Low-Fidelity (LF)

Medium-Fidelity (MF)

High-Fidelity (HF)

June 21-24, 2003
Direct FE analysis

Deflection

Convergence of

\((w, \theta_y)_{\text{point A}}\)

Norm. Displ. @ A

\(\text{dof}\)

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Inverse FEM using 1-1 mapping of strains from MIN3 analysis

\[ \varepsilon^\delta_i \text{ @ centroids} \]

\[ \text{MIN3} \]

\[ \lambda = 10^{-6} \]

MIN3

- **Exact reconstruction** of

\[ (u, v, w, \theta_x, \theta_y) \]

for all meshes
Inverse FEM using n-1 mapping of strains from MIN3 analysis

Direct FEM

\text{MIN3}

\varepsilon_i^{\delta} @ centroids

Inverse FEM

\text{MIN3}^{-1}

Lambda study

% Error in w, \theta_y

MAFELAP-2003
Inverse FEM

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Strains with Random Error

\[ \varepsilon^\delta = \varepsilon^* (1 + 0.05 \delta), \]
\[ \delta \in [-1, 1] \text{ (pseudorandom number)} \]
Remarks on Numerical Results

- $\text{MIN}3^{-1}$ exact inverse of $\text{MIN}3$
  - Perfect correspondence in 1-1 mapping
  - Strains sampled at element centroids
- No thin-regime limitations
- Lambda parameter ensures robustness
  - Serves as filter/smooth for $n$-1 mapping
  - Strains with random error "smoothed" effectively
Computational-experimental validation

- Aluminum plate, 2024-T3 alloy
  - Elastic properties: $E=10.6$ Msi, $n=0.33$
  - Dimensions: 10"x3"x1/8"
- Force applied at (9", 1.5")
  - $P = 5.784$ lb (2623 g)

All dimensions in inches
Experiment

DCDT measuring deflection @ (8/7/16, 1 1/2)

Plate instrumented with 28 strain rosettes and DCDT
FE Modeling of Experiment

Triangular Element Meshes

Low Fidelity (LF)
- 28 elements
- 24 nodes

Medium Fidelity (MF)
- 216 elements
- 133 nodes

High Fidelity (HF)
- 864 elements
- 481 nodes
FEM convergence study of tip deflection

- Linear response
  - differences between linear and nonlinear solutions less than 0.01%
Comparison of Deflection

Direct FEM
High-Fidelity Mesh
Max. W = 0.2699 in

Inverse FEM
$\varepsilon^g$ from test
Low-Fidelity Mesh
Max. W = 0.2701 in

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Summary

Variational principle and robust inverse FEM developed for full-field displacement reconstruction from measured strains

- Thin and moderately thick
  - Beams, plates and flat shells
  - Linear response

- Method inherently regularized
  - Strain-displacement relations enforced
  - Integrability (strain compatibility) conditions fulfilled
  - Strain-sensor location and mesh/interpolation dependency

- Standard FEM architecture
  - Accommodates complex structures
  - Independent of material properties
  - Computationally efficient

MAFELAP-2003
Inverse FEM

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Summary (cont.)

- Superior reconstruction quality
  - Computational validation
  - Computational-Experimental validation
- Future extension to
  - Large displacement response
  - Non-collocated strain measurements
  - Full-field reconstruction of strains, stresses, and failure criteria
  - Curved shells
  - Built-up structures